Phys 410
Spring 2013
Lecture #23 Summary
15 March, 2013

The rotating bead on a loop problem was analyzed. A bead of mass m is constrained to move on a vertical circular loop of radius R, and the loop is set into rotation about the vertical axis through the loop center, at angular frequency ω. There is a single generalized coordinate θ, which is the angle that the bead makes with respect to the vertically-down direction from the center of the loop. There are two components of velocity for the bead, one around the loop ($v_\theta = R\dot{\theta}$) and the other around the vertical axis ($v_\varphi = \rho\omega = R\sin\theta\,\omega$). The Lagrangian is $\mathcal{L}(\theta,\dot{\theta}) = T - U = \frac{mR^2}{2}\left(\dot{\theta}^2 + \omega^2\sin\theta^2\right) - mgR(1-\cos\theta)$. The resulting equation of motion is $\ddot{\theta} = (\omega^2\cos\theta - g/R)\sin\theta$. This cannot be solved in closed form. Note that the equation reduces to the equation of motion of a pendulum in the limit $\omega \to 0$.

Even though we cannot solve this equation, we can learn much about the motion. From the in-class demonstration we showed that there are several different equilibrium points for the bead while the loop is rotating. The equilibrium points are those special angles θ_0 where a particle can be placed with no initial velocity $\dot{\theta} = 0$ and will stay there because the acceleration is zero, $\ddot{\theta} = 0$. The zeroes of the above equation of motion come from the two terms in the product on the RHS. The first are those for which $\sin\theta_0 = 0$, which include $\theta_0 = 0, \pi$. The position $\theta_0 = \pi$ is always unstable, while that for $\theta_0 = 0$ is stable for low angular velocities ω. The other equilibrium points are given by the zero of the term in parentheses: $\cos\theta_0 = g/\omega^2 R$. However, since the magnitude of $\cos\theta_0$ is bounded, this requires a certain minimum angular velocity, or greater, to be satisfied: $\omega \geq \sqrt{g/R}$. There are two equilibrium angles in this case: $\theta_0 = \pm\cos^{-1}(g/\omega^2 R)$, both of which are stable when they exist. In summary, the angle $\theta_0 = 0$ is stable for $\omega < \sqrt{g/R}$, and it bifurcates (becomes unstable) into two other stable points at $\theta_0 = \pm\cos^{-1} g/\omega^2 R$. In the limit as $\omega \to \infty$, the angles become $\theta_0 = \pm\pi/2$, which is the 'outside' of the circular hoop.

If a generalized coordinate does not appear in the Lagrangian it is said to be ignorable or cyclic. The corresponding generalized momentum is conserved.

Finally we derived a new quantity known as the Hamiltonian. The Lagrangian was engineered specifically to reproduce Newton's second law in component form, however it does not have a simple physical interpretation. By taking the total time derivative of the Lagrangian we could create a new quantity $\mathcal{H}$ that is time-invariant, subject to the condition that $\frac{\partial\mathcal{L}}{\partial t} = 0$, and it is found to be $\mathcal{H} = \sum_{i=1}^{n} p_i\,\dot{q}_i - \mathcal{L}$, where $p_i = \partial\mathcal{L}/\partial\dot{q}_i$. If, in addition, there is a time-independent relationship between the Cartesian coordinates and the generalized coordinates, $\vec{r}_\alpha = \vec{r}_\alpha(q_1, q_2, \dots q_i, \dots q_n)$, then the Hamiltonian has a simple interpretation as the total mechanical energy $T + U$.